Invariant Functions in Nonrelativistic Theory

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Galilean invariance is applied to extract the invariant parts of matrix elements of particle creation and annihilation operators. A method for solving the N-particle problem in terms of N to $N+1$ particle amplitudes is presented.

1. INTRODUCTION

THE usual treatment of the N-particle problem deals with the full N-particle wave function $\Psi(\mathbf{x}_1,\cdots,\mathbf{x}_N)$. A recent alternative is the approach via Green's functions,[1] which emphasizes the N to $N+1$ particle amplitudes. However, in order to treat the N to $N+1$ particle amplitudes, the Green's-function methods also require N to $N+2$, N to $N-1$, N to $N-2$, etc., particle amplitudes. This paper presents a formulation of the $N+1$ particle problem as far as possible in terms of N to $N+1$ particle amplitudes alone. As will be seen, the other amplitudes which will be required will all involve fewer than $N+1$ particles.

In order to simplify the treatment, the system is assumed to consist of identical spinless particles; the Hamiltonian is (in units such that $\hbar=2m=1$)

$$E=\int \nabla\psi^\dagger(\mathbf{x})\cdot\nabla\psi(\mathbf{x})d\mathbf{x} +\frac{1}{2}\int V(\mathbf{x}-\mathbf{y})\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x})d\mathbf{x}d\mathbf{y}, \quad (1)$$

where $\psi(\mathbf{x})$, $\psi^\dagger(\mathbf{x})$ are the second-quantized field operators that satisfy the (anti)commutation relations

$$[\psi(\mathbf{x}),\psi(\mathbf{y})]_\pm=0,\quad [\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})]_\pm=\delta(\mathbf{x}-\mathbf{y}). \quad (2)$$

Here, as in the following, the upper (lower) sign is for fermions (bosons).

The N to $N+1$ particle amplitudes of interest are the amplitudes $\langle N\alpha\mathbf{K}|\psi(x)|N+1\,\beta\mathbf{K}'\rangle$, where $|N\alpha\mathbf{K}\rangle$ is the state consisting of N particles in internal state α with total (center-of-mass) momentum $\mathbf{K}$. The differential equations satisfied by these amplitudes, or, more exactly, by their invariant parts, defined in Sec. 2, are derived in Sec. 3. The main problem is ensuring the (anti)symmetry of the system with respect to interchange of particles. Since the amplitudes may contain as few as one coordinate, this cannot be achieved by the usual (anti)symmetrization procedure in the particle coordinates. Sections 4 and 5 discuss the resolution of this problem for continuum states and for bound states, respectively, and thus complete the formulation of the $(N+1)$-particle problem in terms of the invariant N to $N+1$ particle amplitudes. The final section considers the determination of the T matrix from the invariant amplitudes.

2. INVARIANT FUNCTIONS

Before deriving the differential equations satisfied by the amplitudes, it is useful to extract the invariant parts of the amplitudes by removing the trivial part of the dependence on $\mathbf{K}$, $\mathbf{K}'$, and $\mathbf{x}$. First, it follows from translation invariance that

$$\psi(\mathbf{x})=e^{-i\mathbf{P}\cdot\mathbf{x}}\psi(0)e^{i\mathbf{P}\cdot\mathbf{x}}, \quad (3)$$

where $\mathbf{P}$ is the total momentum operator. Therefore,

$$\langle N\alpha\mathbf{K}|\psi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle =e^{i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{x}}\langle N\alpha\mathbf{K}|\psi(0)|N+1\,\beta\mathbf{K}'\rangle. \quad (4)$$

Then Galilean invariance requires that

$$\langle N\alpha\mathbf{K}|\psi(0)|N+1\,\beta\mathbf{K}'\rangle$$

depend on $\mathbf{K}$ and $\mathbf{K}'$ only in the combination corresponding to the relative velocity, namely, $\mathbf{K}'/(N+1)-\mathbf{K}/N$. Specifically, the choice of variable that will be used is

$$\langle N\alpha\mathbf{K}|\psi(0)|N+1\,\beta\mathbf{K}'\rangle=(2\pi)^{-3/2}\tilde{\psi}_{\alpha\beta}\left(\frac{N}{N+1}\mathbf{K}'-\mathbf{K}\right), \quad (5)$$

so that

$$\langle N\alpha\mathbf{K}|\psi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle =(2\pi)^{-3/2}e^{i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{x}}\tilde{\psi}_{\alpha\beta}\left(\frac{N}{N+1}\mathbf{K}'-\mathbf{K}\right). \quad (6)$$

The functions $\tilde{\psi}_{\alpha\beta}$ are clearly invariant under translations in both configuration space and velocity space, and will be called the invariant N to $N+1$ particle amplitudes. Application of standard group-theoretic techniques shows that, if α and β belong to rotation-group representations J_α and J_β and correspond to the elements M_α and M_β of these representations, respectively, then $\psi_{\alpha\beta}(\mathbf{k})$ can have components corresponding to the $M_\beta-M_\alpha$ elements of the representations $|J_\alpha-J_\beta|$, $|J_\alpha-J_\beta|+1$, $\cdots$, $J_\alpha+J_\beta$.

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[1] A. Klein and R. Prange, Phys. Rev. **112**, 994 (1958). V. M. Galitskii and A. B. Migdal, Zh. Eksperim. i. Teor. Fiz. **34**, 139 (1958) [translation: Soviet Phys.—JETP **7**, 96 (1958)].

Note that (6) gives

$$\psi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle$$
$$=S_\alpha\int d\mathbf{K}\,|N\alpha\mathbf{K}\rangle\langle N\alpha\mathbf{K}|\psi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle$$
$$=(2\pi)^{-3/2}S_\alpha\int d\mathbf{K}\,|N\alpha\mathbf{K}\rangle \times\tilde{\psi}_{\alpha\beta}\left(\frac{N}{N+1}\mathbf{K}'-\mathbf{K}\right)e^{i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{x}}, \quad (7)$$

where the normalization of the states is chosen so that the completeness relation for physical states γ is

$$S_\gamma\int d\mathbf{K}\,|N\gamma\mathbf{K}\rangle\langle N\gamma\mathbf{K}|=\mathbf{1}_N, \quad (8)$$

with S_γ a generalized sum (integral) over internal states γ and $\mathbf{1}_N$ the unit operator in the N-particle subspace. The usual $N+1$ particle wave function for the state $|N+1\,\beta\mathbf{K}'\rangle$ is

$$\Psi_{\beta\mathbf{K}'}{}^{N+1}(\mathbf{x}_1\cdots\mathbf{x}_{N+1})=[(N+1)!]^{-1/2} \times\langle 0|\psi(\mathbf{x}_{N+1})\cdots\psi(\mathbf{x}_1)|N+1\,\beta\mathbf{K}'\rangle, \quad (9)$$

so that (7) gives

$$\Psi_{\beta\mathbf{K}'}{}^{N+1}(\mathbf{x}_1\cdots\mathbf{x}_{N+1})$$
$$=(N+1)^{-1/2}(2\pi)^{-3/2}S_\alpha\int d\mathbf{K}\,e^{i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{x}_1} \times\tilde{\psi}_{\alpha\beta}\left(\frac{N}{N+1}\mathbf{K}'-\mathbf{K}\right)\Psi_{\alpha\mathbf{K}}{}^N(\mathbf{x}_2\cdots\mathbf{x}_{N+1}). \quad (10)$$

Equation (10) exhibits the explicit dependence on $\mathbf{x}_1$ of the functions $\varphi_{\gamma\delta}(\mathbf{x}_1)$ in the common expansion[2]

$$\Psi_\delta{}^{N+1}(\mathbf{x}_1\cdots\mathbf{x}_{N+1})=S_\gamma\Psi_\gamma{}^N(\mathbf{x}_2\cdots\mathbf{x}_{N+1})\varphi_{\gamma\delta}(\mathbf{x}_1). \quad (11)$$

As is evident from (10), it is not the dependence of $\varphi_{\gamma\delta}(\mathbf{x}_1)$ on $\mathbf{x}_1$ that is interesting, but rather the dependence of its invariant part $\tilde{\psi}$ on the velocity difference $\mathbf{v}_\delta-\mathbf{v}_\gamma$ and on the internal states in γ and δ.

The invariant amplitudes have a definite normalization:

$$\langle N+1\,\alpha\mathbf{K}|\int\psi^\dagger(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}|N+1\,\beta 0\rangle =(N+1)\delta(\mathbf{K})\delta_{\alpha,\beta}{}^{(N+1)}, \quad (12)$$

where

$$\langle N+1\,\alpha\mathbf{K}|N+1\,\beta 0\rangle=\delta(\mathbf{K})\delta_{\alpha,\beta}{}^{(N+1)}, \quad (13)$$

and Eq. (8) define $\delta_{\alpha\beta}{}^{(N+1)}$. It follows that

$$(N+1)\delta_{\alpha\beta}{}^{(N+1)}\delta(\mathbf{K})$$
$$=\int d\mathbf{x}\,S_\gamma\int d\mathbf{K}'\,\langle N+1\,\alpha\mathbf{K}|\psi^\dagger(\mathbf{x})|N\gamma\mathbf{K}'\rangle \times\langle N\gamma\mathbf{K}'|\psi(\mathbf{x})|N+1\,\beta 0\rangle$$
$$=\delta(\mathbf{K})S_\gamma\int d\mathbf{K}'\,\tilde{\psi}_{\gamma\alpha}{}^*(\mathbf{K}')\tilde{\psi}_{\gamma\beta}(\mathbf{K}'), \quad (14)$$

so that

$$S_\gamma\int d\mathbf{p}\,\tilde{\psi}_{\gamma\alpha}{}^*(\mathbf{p})\tilde{\psi}_{\gamma\beta}(\mathbf{p})=(N+1)\delta_{\alpha\beta}{}^{(N+1)}. \quad (15)$$

Equation (15) can also be obtained from Eq. (10).

3. DIFFERENTIAL EQUATIONS

The equations for the invariant functions are obtained by taking the matrix elements of

$$[E,\psi(\mathbf{x})]=\nabla^2\psi(\mathbf{x})-\chi(\mathbf{x}), \quad (16)$$

which follows from (1) and (2) with

$$\chi(\mathbf{x})=\int V(\mathbf{x}-\mathbf{y})n(\mathbf{y})\psi(\mathbf{x})d\mathbf{y}, \quad (17)$$

$$n(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x}). \quad (18)$$

It would also be possible to apply the usual Hamiltonian[2] to (7) and (8) and derive the same equations, but invariance arguments are much simpler to apply in the second-quantized formalism. In the matrix elements of (16), the left-hand side is evaluated by using

$$E|N\alpha\mathbf{K}\rangle=(\mathcal{E}_\alpha+N^{-1}K^2)|N\alpha\mathbf{K}\rangle, \quad (19)$$

where $\mathcal{E}_\alpha$ is the internal energy of the state α. For example, $\mathcal{E}_a$ is $-B_a$ for the bound state a, where $B_a>0$ is the binding energy of the state, and $\mathcal{E}_\alpha$ is $[(N+1)/N]p^2-B_b$ for the state consisting asymptotically of N-particle bound state b plus a single particle with relative momentum $\mathbf{p}$ (relative momentum is defined as the product of reduced mass and relative velocity).

By arguments like those leading to (6), it follows that

$$\langle N\alpha\mathbf{K}|\chi(\mathbf{x})|N+1\,\beta\mathbf{K}'\rangle =(2\pi)^{-3/2}e^{i(\mathbf{K}'-\mathbf{K})\cdot\mathbf{x}}\tilde{\chi}_{\alpha\beta}\left(\frac{N}{N+1}\mathbf{K}'-\mathbf{K}\right), \quad (20)$$

where $\tilde{\chi}_{\alpha\beta}$ is the invariant part of the matrix element. The invariant function $\tilde{\chi}$ is related to $\tilde{\psi}$ through

$$\tilde{\chi}_{\alpha\beta}(\mathbf{k})=(2\pi)^{3/2}\langle N\alpha-\mathbf{k}|\chi(0)|N+1\,\beta 0\rangle$$
$$=(2\pi)^{3/2}S_\gamma\int d\mathbf{y}d\mathbf{K}\,V(-\mathbf{y})\langle N\alpha-\mathbf{k}|n(\mathbf{y})|N\gamma\mathbf{K}\rangle \times\langle N\gamma\mathbf{K}|\psi(0)|N+1\,\beta 0\rangle. \quad (21)$$

[2] For example, N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), Chap. VIII.

Now it again follows from Galilean invariance that

$$\langle N\alpha-\mathbf{k}|n(\mathbf{y})|N\gamma\mathbf{K}\rangle=(2\pi)^{-3/2}e^{i(\mathbf{k}+\mathbf{K})\cdot\mathbf{y}}\,\tilde{n}_{\alpha\gamma}(\mathbf{K}+\mathbf{k}),\quad(22)$$

so that

$$\tilde{\chi}_{\alpha\beta}(\mathbf{k})=(2\pi)^{-3/2}S_\gamma\int d\mathbf{K}d\mathbf{y}\,e^{-i\mathbf{K}\cdot\mathbf{y}}V(\mathbf{y})\tilde{n}_{\alpha\gamma}(\mathbf{K})\tilde{\psi}_{\gamma\beta}(\mathbf{k}-\mathbf{K})$$
$$=S_\gamma\int d\mathbf{K}\,\tilde{V}(\mathbf{K})\tilde{n}_{\alpha\gamma}(\mathbf{K})\tilde{\psi}_{\gamma\beta}(\mathbf{k}-\mathbf{K}),\quad(23)$$

where the Fourier-transform convention

$$f(\mathbf{x})=(2\pi)^{-3/2}\int\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\quad(24)$$

will be used for V, $n_{\alpha\gamma}$, $\psi_{\alpha\beta}$, etc.

Hence, the matrix element of (16) between the states $|N\alpha-(N+1)\mathbf{k}\rangle$ and $|N+1\,\beta-(N+1)\mathbf{k}\rangle$ gives

$$\left[\frac{(N+1)k^2}{N}+\mathcal{E}_\alpha-(N+1)k^2-\mathcal{E}_\beta\right]\tilde{\psi}_{\alpha\beta}(\mathbf{k})$$
$$=-S_\gamma\int d\mathbf{K}\,\tilde{V}(\mathbf{K})\tilde{n}_{\alpha\gamma}(\mathbf{K})\tilde{\psi}_{\gamma\beta}(\mathbf{k}-\mathbf{K}),\quad(25)$$

or

$$\left(\frac{N+1}{N}k^2+\mathcal{E}_\alpha-\mathcal{E}_\beta\right)\tilde{\psi}_{\alpha\beta}(\mathbf{k})$$
$$+S_\gamma\int d\mathbf{K}\,\tilde{V}(\mathbf{K})\tilde{n}_{\alpha\gamma}(\mathbf{K})\tilde{\psi}_{\gamma\beta}(\mathbf{k}-\mathbf{K})=0.\quad(26)$$

These equations are the invariant parts of the usual[2] equations for the functions $\varphi_{\gamma\delta}(\mathbf{x})$ of Eq. (11). The Fourier transform of (26) is

$$\left(-\frac{N+1}{N}\nabla^2+\mathcal{E}_\alpha-\mathcal{E}_\beta\right)\psi_{\alpha\beta}(\mathbf{x})$$
$$+S_\gamma\int d\mathbf{y}\,V(\mathbf{x}-\mathbf{y})n_{\alpha\gamma}(\mathbf{y})\psi_{\gamma\beta}(\mathbf{x})=0.\quad(27)$$

Here, $n_{\alpha\gamma}(\mathbf{y})$ is the invariant part of the density matrix element and represents the density at displacement $\mathbf{y}$ from the center of mass of the N-particle system. Note that $n_{\alpha\gamma}$ involves only the N-particle states, and not those of $N+1$ particles. The combination

$$v_{\alpha\gamma}(\mathbf{x})=\int d\mathbf{y}\,V(\mathbf{x}-\mathbf{y})n_{\alpha\gamma}(\mathbf{y})\quad(28)$$

plays the role of the potential in (27). It clearly represents the invariant potential, that is, the potential at displacement $\mathbf{x}$ from the N-particle center of mass. With (28), Eq. (27) takes the form

$$\left(-\frac{N+1}{N}\nabla^2+\mathcal{E}_\alpha-\mathcal{E}_\beta\right)\psi_{\alpha\beta}(\mathbf{x})+S_\gamma v_{\alpha\gamma}(\mathbf{x})\psi_{\gamma\beta}(\mathbf{x})=0.\quad(29)$$

Equations (26) and (29) are clearly invariant under translations in configuration and velocity space, while transforming in the standard way under rotations.

In order to solve Eqs. (29) for the invariant amplitudes $\psi_{\alpha\beta}(\mathbf{x})$, it is necessary to know the two-body interaction $V(\mathbf{x})$, the invariant N-particle density matrix elements $n_{\alpha\gamma}(\mathbf{x})$, the energies $\mathcal{E}_\alpha$ of the N-particle states, and those of the energies $\mathcal{E}_\beta$ which are continuum energies. These latter are composed of binding energies of groups of fewer than $N+1$ particles and relative kinetic energies, and hence do not involve unknown quantities associated with the $(N+1)$-particle system. Clearly, the binding energies of all numbers of particles less than $N+1$ are needed. The energies $\mathcal{E}_\beta$ that are binding energies of $(N+1)$-particle bound states are to be determined as eigenvalues of (29).

4. (ANTI)SYMMETRY IN CONTINUUM STATES

The amplitudes $\psi_{\alpha\beta}(\mathbf{x})$ corresponding to a physical state β must be a solution of (29). In addition, they must satisfy conditions which ensure that the solution has proper characteristics with respect to interchange of particle coordinates, that is, the left-hand side of Eq. (10) must be (anti)symmetric under interchange of particle coordinates. The amplitudes $\psi_{\alpha\beta}$ themselves do not contain all the individual particle coordinates, so that the usual (anti)symmetrization procedure cannot be applied directly to them.

For continuum states it is clear that the asymptotic $|\mathbf{x}|\to\infty$ form of the solution of (29) determines the solution uniquely. Therefore, if a solution of (29) is asymptotically equal to the amplitude for a physical state β, then it will satisfy all the requirements of (anti)symmetry. For continuum states it is thus only necessary to specify exactly the plane-wave part of $\psi_{\alpha\beta}(\mathbf{x})$, or, equivalently, the delta-function part of $\tilde{\psi}_{\alpha\beta}(\mathbf{k})$. The plane-wave part can be found by using Eq. (10) as a starting point, but, again, invariance is easier to apply in the second-quantized formalism.

As an example, let $|N+1\,\beta\mathbf{K}'\rangle$ be $|N+1-n,a,\mathbf{k}_a';n,b,\mathbf{k}_b'^{\text{in}}\rangle$, where a and b are bound states of $N+1-n$ and n particles, respectively, and $N+1-n$, a is not the same as n, b [if the two are the same, the normalization of the state has an extra $2^{-1/2}$, owing to (8)]. Then $\mathbf{K}'=\mathbf{k}_a'+\mathbf{k}_b'$ and the relative momentum in state β is $n(N+1-n)(N+1)^{-1}[n^{-1}\mathbf{k}_a'-(N+1-n)^{-1}\mathbf{k}_b']$. It is convenient to choose $\mathbf{k}_b'=-\mathbf{k}_a'=\mathbf{k}$, $\mathbf{K}'=0$, so that β is $(N+1-n,a;n,b;\mathbf{k}^{\text{in}})$. The delta-function contributions to

$$\langle N\alpha-\mathbf{p}|\psi(0)|N+1\,\beta0\rangle=(2\pi)^{-3/2}\tilde{\psi}_{\alpha\beta}(\mathbf{p})\quad(30)$$

come when $|N\alpha-\mathbf{p}\rangle$ is either

$$\left|N+1-n,\, a,\, -\mathbf{q}-\frac{N+1-n}{N}\mathbf{p};\;\; n-1,\, \gamma^{\text{in}},\, \mathbf{q}-\frac{n-1}{N}\mathbf{p}^{\text{in}}\right\rangle$$

or

$$\left|N-n,\, \delta^{\text{in}},\, -\mathbf{q}-\frac{N-n}{N}\mathbf{p};\;\; n,\, b,\, \mathbf{q}-\frac{n}{N}\mathbf{p}^{\text{in}}\right\rangle,$$

where γ^{in} and δ^{in} are arbitrary "in" states of $n-1$ and $N-n$ particles, respectively, and the momenta have been chosen so that the relative momentum in the state α is $\mathbf{q}$ in both cases. In the first case, α is $(N+1-n, a;\; n-1, \gamma^{\text{in}};\; \mathbf{q}^{\text{in}})$ and the delta-function part of (30) is

$$\delta\left(\mathbf{k}-\mathbf{q}-\frac{N+1-n}{N}\mathbf{p}\right)\left\langle n-1\,\gamma^{\text{in}}\,\mathbf{q}-\frac{n-1}{N}\mathbf{p}\,|\psi(0)|\,nb\mathbf{k}\right\rangle$$
$$=(2\pi)^{-3/2}\left(\frac{N}{N+1-n}\right)^3\delta\left(\mathbf{p}+\frac{N}{N+1-n}(\mathbf{q}-\mathbf{k})\right)$$
$$\times\tilde{\psi}_{\gamma^{\text{in}}b}\left(\frac{n-1}{n}\mathbf{k}-\mathbf{q}+\frac{n-1}{N}\mathbf{p}\right),\quad(31)$$

so that

$$\psi_{\alpha\beta}(\mathbf{x})\rightarrow(2\pi)^{-3/2}\left(\frac{N}{N+1-n}\right)^3 e^{i[N/(N+1-n)](\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}$$
$$\times\tilde{\psi}_{\gamma^{\text{in}}b}\left(\frac{(n-1)(N+1)}{n(N+1-n)}\mathbf{k}-\frac{N}{N+1-n}\mathbf{q}\right)$$
$$+\text{outgoing spherical waves.}\quad(32)$$

Similarly, in the second case, α is $(N-n, \delta^{\text{in}};\; n, b;\; \mathbf{q}^{\text{in}})$ and

$$\psi_{\alpha\beta}(\mathbf{x})\rightarrow(\mp)^n(2\pi)^{-3/2}\left(\frac{N}{n}\right)^3 e^{i(N/n)(\mathbf{q}-\mathbf{k})\cdot\mathbf{x}}$$
$$\times\tilde{\psi}_{\delta^{\text{in}}a}\left(\frac{N}{n}\mathbf{q}-\frac{(N-n)(N+1)}{n(N+1-n)}\mathbf{k}\right)$$
$$+\text{outgoing spherical waves.}\quad(33)$$

For all α other than these two, $\psi_{\alpha\beta}(\mathbf{x})$ has only outgoing spherical waves at $|\mathbf{x}|\rightarrow\infty$. For $n=1$, $N\neq1$, (32) becomes, with $\beta=(N, a;\; 1;\; \mathbf{k}^{\text{in}})$ and $\alpha=(N,a)$

$$\psi_{\alpha\beta}(\mathbf{x})\rightarrow(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}+\text{outgoing spherical waves.}\quad(34)$$

Equations (32)–(34) **give** the invariant part of the usual[2] asymptotic boundary conditions on the $\varphi_{\gamma\delta}(\mathbf{x})$ of Eq. (11).

The plane-wave part of other continuum states can be found in a similar way. Thus, the continuum invariant functions $\psi_{\alpha\beta}(\mathbf{x})$ are completely specified by the differential equations (29) together with asymptotic boundary conditions of the form of (32)–(34). Note that the form factors $\tilde{\psi}$ on the right-hand side of (32) and (33) involve only systems containing fewer than $N+1$ particles. These form factors are required in order to determine the $\psi_{\alpha\beta}(\mathbf{x})$ for continuum states β.

5. (ANTI)SYMMETRY IN BOUND STATES

In order to distinguish the physical bound-state solutions of (29) from the unphysical ones, it is clear that (anti)symmetry and completeness must be imposed via the commutation relations (2) and the completeness relation (8) for the physical states:

$$\left\langle N\alpha\frac{\mathbf{K}}{2}\right|[\psi(\mathbf{y}/2),\,\psi^\dagger(-\mathbf{y}/2)]_\pm\left|N\beta-\frac{\mathbf{K}}{2}\right\rangle$$
$$=\delta(\mathbf{y})\delta(\mathbf{K})\delta_{\alpha\beta}{}^{(N)}$$
$$=S_\gamma\int d\mathbf{K}'\,\langle N\alpha\,\mathbf{K}/2|\psi(\mathbf{y}/2)|N+1\,\gamma\mathbf{K}'\rangle$$
$$\times\langle N+1\,\gamma\mathbf{K}'|\psi^\dagger(-\mathbf{y}/2)|N\beta\,-\mathbf{K}/2\rangle$$
$$\pm S_\delta\int d\mathbf{K}'\,\langle N\alpha\,\mathbf{K}/2|\psi^\dagger(-\mathbf{y}/2)|N-1\,\delta-\mathbf{K}'\rangle$$
$$\times\langle N-1\,\delta-\mathbf{K}'|\psi(\mathbf{y}/2)|N\beta\,-\mathbf{K}/2\rangle$$
$$=(2\pi)^{-3}\int d\mathbf{K}'\,e^{i\mathbf{K}'\cdot\mathbf{y}}$$
$$\times\left\{S_\gamma\tilde{\psi}_{\alpha\gamma}\left(\frac{N}{N+1}\mathbf{K}'-\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\beta\gamma}{}^*\left(\frac{N}{N+1}\mathbf{K}'+\frac{\mathbf{K}}{2}\right)\right.$$
$$\left.\pm S_\delta\tilde{\psi}_{\delta\alpha}{}^*\left(\mathbf{K}'+\frac{N-1}{N}\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\delta\beta}\left(\mathbf{K}'-\frac{N-1}{N}\frac{\mathbf{K}}{2}\right)\right\},\quad(35)$$

and, therefore,

$$\delta(\mathbf{K})\delta_{\alpha\beta}{}^{(N)}=S_\gamma\tilde{\psi}_{\alpha\gamma}\left(\frac{N}{N+1}\mathbf{K}'-\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\beta\gamma}{}^*\left(\frac{N}{N+1}\mathbf{K}'+\frac{\mathbf{K}}{2}\right)$$
$$\pm S_\delta\tilde{\psi}_{\delta\alpha}{}^*\left(\mathbf{K}'+\frac{N-1}{N}\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\delta\beta}\left(\mathbf{K}'-\frac{N-1}{N}\frac{\mathbf{K}}{2}\right),\quad(36)$$

where the states γ are the physical $(N+1)$-particle states and the states δ are physical $(N-1)$-particle states. The continuum states γ can be found as described in the preceding section, and are assumed known, as are the $(N-1)$-particle states δ. Therefore, the correct set B of bound-state amplitudes must satisfy

$$\sum_{\gamma\in B}\tilde{\psi}_{\alpha\gamma}\left(\frac{N}{N+1}\mathbf{K}'-\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\beta\gamma}{}^*\left(\frac{N}{N+1}\mathbf{K}'+\frac{\mathbf{K}}{2}\right)$$
$$=\delta(\mathbf{K})\delta_{\alpha\beta}{}^{(N)}$$
$$\mp S_\delta\tilde{\psi}_{\delta\alpha}{}^*\left(\mathbf{K}'+\frac{N-1}{N}\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\delta\beta}\left(\mathbf{K}'-\frac{N-1}{N}\frac{\mathbf{K}}{2}\right)$$
$$-S_{\gamma\in C}\tilde{\psi}_{\alpha\gamma}\left(\frac{N}{N+1}\mathbf{K}'-\frac{\mathbf{K}}{2}\right)\tilde{\psi}_{\beta\gamma}{}^*\left(\frac{N}{N+1}\mathbf{K}'+\frac{\mathbf{K}}{2}\right),\quad(37)$$

where (37) must hold for all α, β, and $\mathbf{K}'$. By choosing a single $\alpha=\beta$ and a set of points $\mathbf{K}_i'$, $i=1, 2, \cdots, n$, with n the total number of bound solutions of (29) (n may be denumerably infinite), Eq. (37) can be used to determine which of the bound solutions belong to the set B of physical solutions. Clearly, this condition is not very practical. The techniques of this paper are more useful for continuum states than for bound states.

With reference to (11), the physical functions $\varphi_{\gamma\delta}$ give Ψ_δ which are (anti)symmetric. Thus, the unphysical solutions $\varphi_{\gamma\delta_u}$ are such that the (anti)symmetrized analog of (11) is zero, that is, if O is the (anti)symmetrization operator, and $\varphi_{\gamma\delta_u}$ corresponds to an unphysical solution of (29), then

$$OS_\gamma\Psi_\gamma(\mathbf{x}_2\cdots\mathbf{x}_{N+1})\varphi_{\gamma\delta_u}(\mathbf{x}_1)=0. \tag{38}$$

In the case of fermions, if Ψ_γ in (38) were a product of single-particle functions, then (38) would be satisfied if $\varphi_{\gamma\delta_u}$ were any of the single-particle functions, so that (38) is the equation which describes the "occupied" single-particle states in fermion systems. For bosons, at least two values of γ must occur in a sum of type (38) in order to produce a zero on the right-hand side. The interpretation of the unphysical bound-state solutions is not clear in this case.

6. SCATTERING AMPLITUDE

From the preceding sections it follows that the continuum amplitudes are determined by the differential equations (29) and associated boundary conditions like (32)–(34), while the bound-state amplitudes are similarly determined by the differential equations (29) and the supplementary condition (37). All that remains is to specify the T matrix in terms of the invariant amplitudes. Only processes in which the initial or final state contains an incident or emerging single particle are easily treated in this framework.

The example considered here is the T matrix element for final state $|N\gamma^{\text{out}}-\mathbf{p};\, 1, \mathbf{p}^{\text{out}}\rangle=|N+1\, f0\rangle$ with $f=(N\gamma^{\text{out}};\, 1;\, \mathbf{p}^{\text{out}})$ representing a single particle emerging from the group making up the asymptotic γ^{out} state. The initial state is taken to be an arbitrary $(N+1)$-particle "in" state $|N+1\, i^{\text{in}}\mathbf{P}\rangle$. Then it follows from the LSZ formalism[3] that

$$\langle N+1\, f^{\text{out}}0|N+1\, i^{\text{in}}\mathbf{P}\rangle=\langle N+1\, f^{\text{in}}0|N+1\, i^{\text{in}}\mathbf{P}\rangle$$
$$-2\pi i\delta(\mathbf{P})\delta(\mathcal{E}_i-\mathcal{E}_f)\langle N+1\, f|T|N+1\, i\rangle, \tag{39}$$

with T given by

$$\langle N\gamma^{\text{out}};\, 1;\, \mathbf{p}|T|N+1\, i\rangle=\tilde{\chi}_{\gamma^{\text{out}}{}_i{}^{\text{in}}}(\mathbf{p}). \tag{40}$$

The operator χ is essentially the source operator for the field, so that (40) is a standard form.

If γ is a bound state, so that $\gamma^{\text{out}}=\gamma^{\text{in}}=\gamma$, then $\chi_{\gamma i^{\text{in}}}(\mathbf{x})$ can be determined from Sec. 4. However, if $\gamma^{\text{out}}\neq\gamma^{\text{in}}$, then Sec. 4 gives $\chi_{\gamma^{\text{in}}{}_i{}^{\text{in}}}(\mathbf{x})$ instead of $\chi_{\gamma^{\text{out}}{}_i{}^{\text{in}}}(\mathbf{x})$, and determination of the latter requires a further transformation:

$$\begin{aligned}
&\tilde{\chi}_{\gamma^{\text{out}}{}_i{}^{\text{in}}}(\mathbf{p})\\
&=(2\pi)^{3/2}\langle N\gamma^{\text{out}}-\mathbf{p}|\chi(0)|N+1\, i^{\text{in}}0\rangle\\
&=(2\pi)^{3/2}S_\alpha\int d\mathbf{K}\,\langle N\gamma^{\text{out}}-\mathbf{p}|N\alpha^{\text{in}}\mathbf{K}\rangle\\
&\qquad\qquad\times\langle N\alpha^{\text{in}}\mathbf{K}|\chi(0)|N+1\, i^{\text{in}}0\rangle\\
&=(2\pi)^{3/2}\langle N\gamma^{\text{in}}-\mathbf{p}|\chi(0)|N+1\, i^{\text{in}}0\rangle\\
&\qquad-2\pi i(2\pi)^{3/2}S_\alpha\delta(\mathcal{E}_\alpha-\mathcal{E}_\gamma)\langle N\gamma|T|N\alpha\rangle\\
&\qquad\qquad\times\langle N\alpha^{\text{in}}-\mathbf{p}|\chi(0)|N+1\, i^{\text{in}}0\rangle\\
&=\tilde{\chi}_{\gamma^{\text{in}}{}_i{}^{\text{in}}}(\mathbf{p})-2\pi iS_\alpha\delta(\mathcal{E}_\alpha-\mathcal{E}_\gamma)\\
&\qquad\qquad\times\langle N\gamma|T|N\alpha\rangle\tilde{\chi}_{\alpha^{\text{in}}{}_i{}^{\text{in}}}(\mathbf{p}).
\end{aligned} \tag{41}$$

Thus, determination of $\tilde{\chi}_{\gamma^{\text{out}}{}_i{}^{\text{in}}}$ requires the $\tilde{\chi}_{\alpha^{\text{in}}{}_i{}^{\text{in}}}$ and the N-particle T matrix. In this formulation, therefore, the $N+1$ particle T matrix involves the N-particle T matrix in an explicit way.

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